Band gaps in the propagation and scattering of surface water waves over cylindrical steps

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Here we investigate the propagation and scattering of surface water waves in the presence of arrays of bottom-mounted cylindrical steps. Both periodic and random arrangements of the steps are considered. The wave transmission through the arrays is computed using the multiple scattering method based upon a recently derived formulation. For the periodic case, the results are compared to the band structure calculation. We demonstrate that complete band gaps can be obtained in such a system. Furthermore, we show that the randomization of the location of the steps can significantly reduce the transmission of water waves. Comparison with other systems is also discussed.

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Phenomena pertinent to waves in complex media have been and continue to be a great inspiration for scientific explorations. One of the most important phenomena is frequency band structures. It prevails when waves propagate through periodically structured media. Such a phenomenon was first investigated for electrons in solids nearly eighty years ago. The well-known Bloch theorem has then been proposed and led to the successful explanation of some important properties of solids such as conductivity, semiconductivity, and insulating states [1]. Applying these concepts to classical waves [2,3] has paved an avenue to the new era of research. Not only the phenomena previously observed or discussed only for electronic systems are successfully transplanted to classical systems, but many more significant and novel ideas and applications, have well gone beyond expectation, and are so far reaching that a fruitful new field has been established, i e., the field of photonic and acoustic crystals.

Recently, the consideration of waves in periodic media has also been deliberately extended to the propagation of water waves over periodically structured bottoms [4–9]. Some of the advances have been reviewed, for example, by McIver [10]. One of the most recent pioneering experiments used surface water waves to illustrate the phenomenon of Bloch waves as a result of the modulation by periodic bottom structures [7]. This experiment made it possible that some abstract concept associated with wave phenomena can be presented in an unprecedentedly clear manner.

Motivated by the experiment described in Ref. [7], in this paper we would like to further explore the propagation of water waves through the underwater structures discussed in the experiment. The structures considered here consist of arrays of cylindrical steps mounted on a flat bottom. There have been many theoretical approaches for investigating propagation of water waves over various bottom topographies, as reviewed in, e.g., Refs. [11–13]. In this paper, we will use the theory, which was first used in Ref. [14] and later was derived in Ref. [15]. The main reason lies in that the theory seems to be useful in explaining the experimental observations on the situations to be considered here (see Ref. [14]). In addition, it has been shown that this approach compares favorably with existing approximations when applied to some special cases considered previously, such as one-step problems [15]. Furthermore, we also applied the theory to the experimental results on one-dimensional systems [23], and recovered the experimental data well; the results will be published elsewhere.

We will calculate the wave transmission and band structures for periodically arranged arrays. Then we will show the effect of positional disorders on the transmission. The results suggest that the phenomenon of complete band gaps by analogy with the photonic crystals is also possible for water waves. The results also suggest that there might be the deaf-band phenomenon for the water waves.

Before continuing, it is worth noting here that from a more general perspective, propagation of water waves over topographical bottoms has been a subject of much research from both practical and theoretical aspects since Lamb [16]. From the practical side, the topic is essential to many important ocean engineering problems such as floating bridges and devices in offshore power stations [10]. A great amount of papers and monographs has been published [17–28]. A comprehensive reference on the topic can be found in two excellent textbooks [11,12].

We wish to further comment on the theory used in the present paper. Due to the complexity of the problem, a few possible effects are not considered in the theory [15,23], including such as nonlinearity and evanescent waves. Since the evanescent waves more likely affect the near field propagation, and may be ignored in the problem discussed in this paper. We point out that since there is no exact or rigorous theory for the situations considered here, certain approximations must be taken so that the problem can be manageable. Depending on the nature of questions, different approximations may be used. A way of verification is to compare with experimental data. The success in applying the theory to experiments [14] provides a justification of using the theory. However, we wish to stress that the theory used here is not exact. It will be next task to compare available theories and sort out the suitability of these theories for various situations

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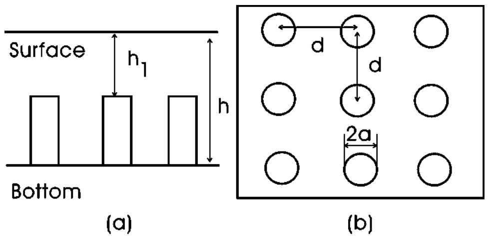

FIG. 1. Conceptual layout of the system: (a) side view and (b) bird's view. Here the cylindrical steps with height $\Delta h = h - h_1$ form a rectangular array with lattice constant d.

and explore possible accumulation effects due to evanescent modes, particularly when the steps [7] are close to each other.

A conceptual layout of the system considered here is presented in Fig. 1. The cylindrical steps with radius a and height $h-h_1$ are placed vertically on a bottom. The steps can be arranged either regularly or randomly. Here the steps form a square lattice with the lattice constant d. The water surface is in the x-y plane. We consider how these steps affect the propagation of the surface waves. This is a two-dimensional problem. The governing equations for the motion of surface water waves in the system described by Fig. 1 can be obtained by invoking the Newton's second law and the conservation of mass by assuming that the water is incompressible. The formulation has been given in Ref. [14] and the detailed derivation is given in Ref. [15], here we just list the final equations.

The displacement of the water surface is denoted by $\eta(\vec{r},t)$. Its Fourier transformation is

$$\eta(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\eta_\omega(\vec{r}). \tag{1}$$

The equation of motion for the Fourier component η_ω is derived as [15]

$$\nabla\left(\frac{1}{k^2}\nabla\ \eta_\omega(\vec{r})\right) + \eta_\omega(\vec{r}) = 0, \tag{2}$$

where $\vec{\nabla} = \partial_x \vec{e}_x + \partial_y \vec{e}_y$, and the wave number k satisfies

$$\omega^2 = gk(\vec{r})\tanh[k(\vec{r})h(\vec{r})]. \tag{3}$$

For a fixed frequency ω, the wave number varies as a function of the depth $h(\vec{r})$.

In this paper we will apply Eq. (2) to case of the cylindrical steps depicted in Fig. 1, in line with the experiment [7]. Furthermore, we assume that all the steps are identical. When there is a stimulating source, the transmitted waves will be scattered repeatedly at the steps, forming an orchestral pattern of multiple scattering. Such a multiple scattering process can be *exactly* solved for any arrangement of the steps by the multiple scattering theory [29]. The wave transmission can be computed. In the computation, the transmission is normalized such that it is unity when there are no scatterers. While the details have been presented in Refs. [15,30], the essence is summarized as follows. The scattered waves from each step is a response to the incident waves which the direct wave from the source and all the scattered waves from other steps. Then express the scattered waves from each step in terms of a series of mode expansion. A set of self-consistent equations is thus obtained, and is solved exactly by a time consuming method of matrix inversion.

When the steps are regularly placed to form periodic lattices, the frequency bands will appear and can be determined as follows. By Bloch's theorem [1], the displacement field η_ω can be expressed in the following form:

$$\eta_\omega(\vec{r}) = e^{i\vec{K}\cdot\vec{r}}\sum_{\vec{G}} C_\omega(\vec{G},\vec{K})e^{i\vec{G}\cdot\vec{r}}, \tag{4}$$

where $\vec{G}$ is the vector in the reciprocal lattice and $\vec{K}$ the Bloch vector [1]. In this case, the wave number k also varies periodically and we have the following expression:

$$\frac{1}{k^2} = \sum_{\vec{G}} A_\omega(\vec{G})e^{i\vec{G}\cdot\vec{r}}. \tag{5}$$

For a fixed ω, the coefficients A_ω are determined from Eqs. (3) and (5).

Substituting Eqs. (4) and (5) into Eq. (2), we get

$$\sum_{\vec{G}'} Q_{\vec{G},\vec{G}'}(\vec{K},\omega)C_\omega(\vec{G}',\vec{K}) = 0, \tag{6}$$

with

$$Q_{\vec{G},\vec{G}'}(\vec{K},\omega) = [(\vec{G}+\vec{K})\cdot(\vec{G}'+\vec{K})]A_\omega(\vec{G}-\vec{G}') - \delta_{\vec{G},\vec{G}'}.$$

The dispersion relation connecting $\vec{K}$ and ω, i.e., the frequency bands, is therefore determined by the secular equation

$$\det[[(\vec{G}+\vec{K})\cdot(\vec{G}'+\vec{K})]A_\omega(\vec{G}-\vec{G}') - \delta_{\vec{G},\vec{G}'}]_{\vec{G},\vec{G}'} = 0. \tag{7}$$

Special care has to be taken in solving the above secular equation, since the initial dispersion relation in Eq. (3) is nonlinear. We use an iterative procedure to find the zero point or the fixed point for the determinant. To gain confidence with the computation, we have applied the numerical codes to three special cases for which the solution is relatively easy to obtain: a flat bottom, steps in shallow and deep waters. We found that the results match well the expectations.

First we consider the case of regular arrays. Figure 2 shows the results for the band structures and the transmission of water waves across the arrays of cylindrical steps. With reference to Ref. [7], the following parameters have been used in the computation: lattice constant d=2.5 mm, cylinder radius a=0.875 mm, depth of the water h=2.5 mm. The heights for the steps are 2.49 and 2.40 mm for (a) and (b), respectively. In the computation, we have also considered the capillary effect by modifying Eq. (3) into $\omega^2 = gk(1+b^2k^2)\tanh(kh)$, and we set the capillary length b as 0.93 mm in accordance with the experiment [7]. When computing the transmission, a stimulating source is placed about one lattice constant away from the arrays whereas the receiver is located at about half of the lattice constant away on the other side of

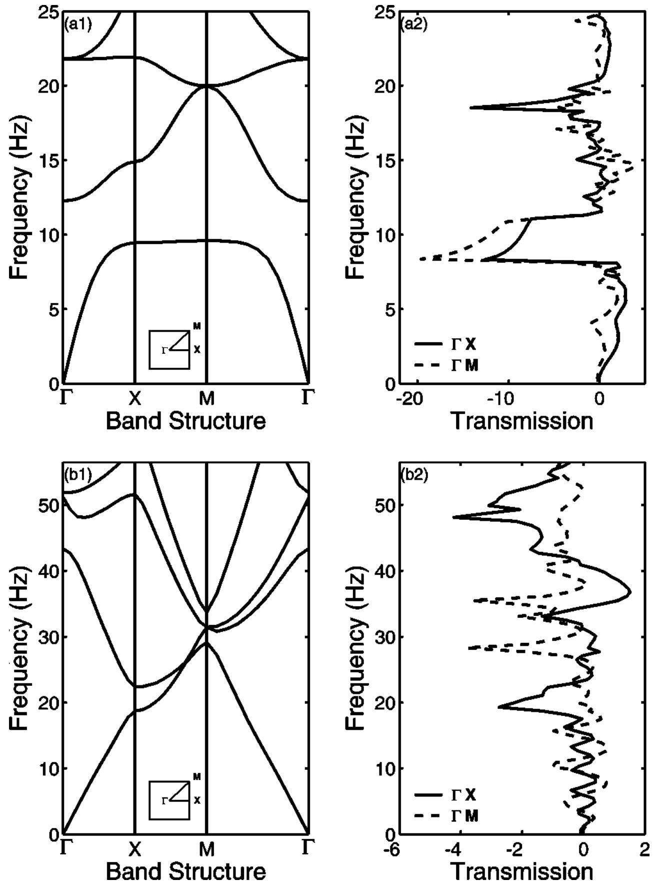

FIG. 2. Right panel: Normalized transmission $\ln|T|^2$ versus frequency for square lattices cylindrical steps with two deferent heights. Left panel: the calculated band structures for the corresponding lattices. The inserted boxes in (a1) and (b1) denote the Brillouin zone and illustrates the direction of wave transmission. For example, ΓX and ΓM refer to 10 and 11 directions, respectively.

the arrays. To ensure the stability of the results, enough modes and number of steps have been considered. For instance, the maximum mode number and the maximum array size considered are 9 and 30×10, respectively. These numbers are considerably larger than what has been computed previously for similar problems.

Here it is shown that there is a complete band gap ranging from 9.6 to 12.3 Hz for the case in (a). The band structure calculation in (a1) is fully supported by the independent transmission calculation by the multiple scattering theory. Along the ΓX direction, the band structure shows that there is a partial frequency gap from about 15 to 22 Hz. That is, waves whose frequencies lie within the range cannot propagate along this direction. This partial gap also appears in the transmission calculation shown by the solid line in Fig. 2(a2). However, the gap depicted by the transmission calculation seems much narrower than that obtained by the band structure calculation. We find that this is due to the finiteness of the array. Since a point source was used to transmit waves, waves can be radiated to various direction. Although the propagation along the ΓX direction is prohibited in the presence of the partial gap, the radiation into other directions may still have the chance to arrive at the receiver, thus complicating the observation of the partial gap.

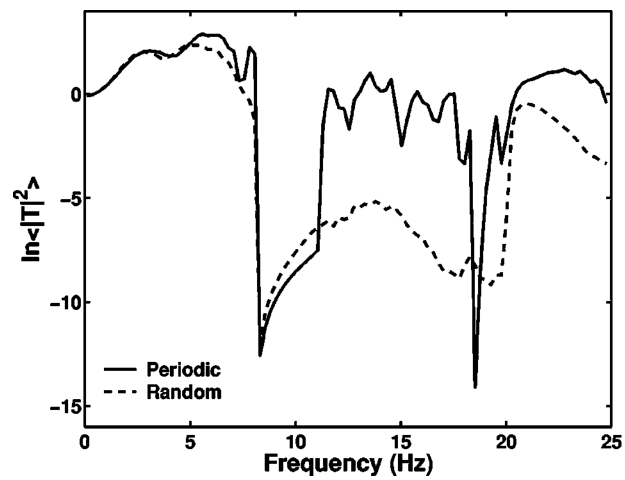

FIG. 3. Normalized transmission $\ln|T|^2$ versus frequency for periodic and complete random arrays, respectively. In the random case, the transmission has been averaged over the random configurations.

We also find that the band structures and the transmission are very sensitive to the arrangement and the height of the steps. As an example, in Fig. 2(b) we show the results for the same lattice array as in (a), except that we change the height of the steps from 2.49 to 2.40 mm. This slight change causes a dramatic change in band structure. The complete band gap disappeared. There are two partial gaps located at 20 and 45 Hz, respectively, along the ΓX direction. Within these two gaps, the transmission is inhibited, as evidenced by the two leftward valleys on the solid line in Fig. 2(b2). Compared to the situation in Fig. 2(a), the transmission data match the band structures better for the partial gaps.

We notice that there are two inhibited transmission valleys along the ΓM direction from the multiple scattering calculation, referring to the dotted line in Fig. 2(b2). This phenomenon is surprising, since in the frequency range concerned, roughly from 28 to 35 Hz, two frequency bands do show up in Fig. 2(b1). A possible explanation for this ambiguity may be that the two bands are deaf. Such a deaf-band phenomenon has been recently observed, for example, in acoustic systems [31] with a further support from theoretical computations [30].

Next we consider the effect of the randomization in the locations of the steps. For brevity, we only show the result for the complete random array. That is, the locations of the cylindrical steps are completely random on the $x-y$ plane. The only restraints are that no two steps should overlap with each other, and the averaged distance between two nearest steps is kept as the same as in the ordered case, i.e., $d=2.5$ mm. The transmission results are shown in Fig. 3. The solid and dotted lines separately refer to the transmission results for the propagation along the [10] direction in the ordered case and for the complete random case. At low frequencies, the disorder effect is not obvious for the given sample size. In this regime, the scattering by the steps is week. However, the transmission is significantly reduced in the mid range of frequency. This observation is in agreement with the case of acoustic scattering by arrays of rigid cylinders located in air [15]. For high frequencies, the reduction due to the disorder is not as significant. This is understandable. In the high frequency range and when the shallow water approximation fails, i.e., when $kh >> 1$, the effect of the steps on the wave propagation tends to diminish. This can be seen from Eq. (3) which reduces to $\omega^2 \approx gk$ in the high frequency regime.

We note that randomness or disorders can lead to the phenomenon of wave localization [32]. This phenomenon has been studied intensively for acoustic waves [33], electromagnetic waves [34], and water waves [23,24]. The water wave localization in other situations has been further investigated recently by a number of groups [28,35]. It was shown that the localization is observable within a range of frequencies for given randomness. The present results for water waves over two dimensional random steps, as shown in Fig. 3, are in qualitative agreement with the previous observations. In addition, effects of nonlinearity on localization have also been studied recently for water waves in other situations [28].

In summary, we have considered water wave propagation over bottom-mounted cylindrical steps. We found that complete band gaps can appear in such a system in analogy with that in the photonic or sonic crystals. The results also suggest that there might be the deaf-band phenomenon for the water waves.

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